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OPTIMAL PURSUIT ON A PLANE IN THE PRESENCE OF AN OBSTACLE*

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A differential game of simple pursuit is examined in the presence of a convex bounded obstacle or arbitrary form. Both players are prohibited from being inside the obstacle and from intersecting its boundary. The statement of the problem by example of a circular obstacle has been given in the book /1/. A complete solution of the encounter game for the circular obstacle case has been constructed by G.K. Pozharitskii (**). A different functional, equal to the players' time of motion up to capture, is used in the present paper; the obstacle's form is not fixed. A method based on the necessary optimality conditions /2,3/ is proposed for the construction of the singular surfaces and the optimal paths in a position pursuit game /4/; the question of the sufficiency of the proposed construction is not examined. Results of a numerical study for two forms of the obstacle are presented.

1. Statement of the problem. In some rectangular coordinate system on a plane let the quantities x_1, x_2 define the coordinates of a point (player) P and x_3, x_4 define the coordinates of a point E. Points P and E have simple isotropic motions, i.e., at each instant can choose arbitrary directions of velocities whose magnitudes do not exceed 1 and ν, respectively, where $0 < \nu < 1$. We introduce into consideration the four-dimensional vectors $x = (x_1, \ldots, x_4)$, $u = (u_1, \ldots, u_4)$, $x, u \in R^4$; the components u_i, $i = 1, 2$ are the controls of player P and u_j, $j = 3, 4$ are the controls of player E. Then the players' equations of motion and the constraints on their controls are written as

$$x^{\cdot} = u, \quad \sqrt{u_1^2 + u_2^2} \leqslant 1, \quad \sqrt{u_3^2 + u_4^2} \leqslant \nu \tag{1.1}$$

On the plane let there be fixed a closed convex bounded set $A, A \subset R^2$, with a piecewise-smooth boundary L. The curve L separates the motion plane into two regions, an external one (which includes the boundary L itself) and an internal one. The latter can be empty only if region A is a segment. An outward normal $n = n(Q) \in R^2$, $Q \in L$ is defined at points Q of boundary L; the function $n(Q)$ is ambiguous if Q is a break point. The players are allowed to move in the external part of the plane and are forbidden to intersect the boundary; motion along the boundary itself is admissible as well. In other words, if player P or E finds himself at a point $Q \in L$, then his controls u_i or u_j must satisfy, in addition to (1.1), the constraints

$$u_1 n_1 + u_2 n_2 \geqslant 0; \ u_3 n_1 + u_4 n_2 \geqslant 0, \ Q \in L \tag{1.2}$$

as well. If the interior part $A - L$ of set A is nonempty, then inequalities (1.2) are a consequence of the phase constraints $P, E \notin A - L$. When A is a segment, $A = L$, the players can in fact be located at any point of the plane; however, at the points of set A they cannot choose arbitrary the direction of motion.

The game starts at instant $t = 0$ and is considered ended at the first instant $t = T > 0$ at which the pursuing point P coincides with the evading point E, i.e., when the inclusion

$$x(T) \in M, \quad M = \{x \in R^4: \ x_1 = x_3, \ x_2 = x_4\} \tag{1.3}$$

is effected. As the payoff or the functional in the game we take the capture time T which player P minimizes and E maximizes. We shall assume that the obstacle does not interfere with observation and that both players exactly know at each instant $0 \leqslant t \leqslant T$ the phase vector $x = x(t)$. In correspondence with this we shall later examine a position differential game /4/ in which the players' control vector is a function of the phase coordinates: $u = u(x)$.

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**) The work was reported in the Institute of Applied Mathematics of the Academy of Sciences of the U.S.S.R at the seminar on the theory of optimal control of motion (see Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No.2, 1979).

The investigation is restricted to only a part X, not the whole, of the game's four-dimensional phase space /1/, which corresponds to the disposition of points $P = (x_1, x_2)$, E (x_3, x_4) in the case when the obstacle lies between the players, i.e., when the segment PE has points in common with set A. Thus, in the pursuit game (1.1)–(1.3) we examine the problem of constructing the game's value and the players' optimal position strategies /4/ for $x \in X$. The set of phase space points for which the segment PE does not have points in common with obstacle A is called the region of direct visibility. In this region the solution of the game being studied can be constructed geometrically with the aid of the players' reachability regions.

2. Necessary optimality condition. We assume that the required game value $V(x)$, $x \in X$ exists, is continuous and is differentiable with respect to all directions in R^4. The derivative of function $V(x)$ with respect to the direction $u = (u_1, \ldots, u_4)$ is for brevity denoted by the symbol of total time derivative $V^{\cdot} = V^{\cdot}(x, u)$. We can obtain the following necessary condition for the game value $V(x)$ /2/:

$$\min_{u_i} \max_{u_j} V^{\cdot}(x, u) \geqslant -1 \geqslant \max_{u_j} \min_{u_i} V^{\cdot}(x, u) \tag{2.1}$$

The extrema are computed under constraints (1.2) and (1.3). At the points of continuous differentiability we have $V^{\cdot} = (p, u)$, $p = V_x \in R^4$, while condition (2.1) turns into the Bellman–Isaacs equation

$$\min_{u_i} \max_{u_j} V^{\cdot} = \max_{u_j} \min_{u_i} V^{\cdot} = F(p) = -1 \tag{2.2}$$

$$F(p) = -\sqrt{p_1^2 + p_2^2} + \nu \sqrt{p_3^2 + p_4^2}$$

The symbol (p, u) denotes the scalar product of the vectors; the vector subscript in V_x is used to denote the vector of partial derivatives with respect to the components of the vector-valued index. The optimal motions in the smoothness domains of $V(x)$ are determined by the equations of the characteristics /1/

$$x^{\cdot} = u^* = F_p, \quad p^{\cdot} = -F_x = 0 \tag{2.3}$$

Here u^* is the players' optimal control vector.

3. Primary solution. It is natural to assume that in some part X_1 of region X, $X_1 \subset X$, the players' optimal motion is along a geodesic line, i.e., the shortest line connecting the players and lying in the plane's external part (Fig.1). Allowing for the fact that A can be a segment,

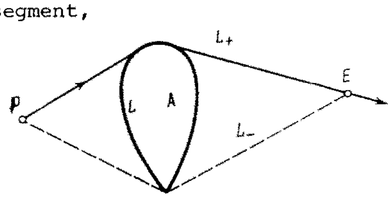

Fig.1

we should give a more precise definiton of a geodesic: constraints (1.2) must not be violated as the players move along it. Together with the geodesic we shall examine extremal lines (extremals) which supply a local minimum of the length of curve PE with due regard to constraints (1.2). Obviously, two extremals L^+ and L^- exist in region X, corresponding to the player P by-passing the obstacle clockwise or counterclockwise (Fig.1). The geodesic line coincides with the extremal of least length. The extremals consist of segments of tangent (support) straight lines to the boundary L and of parts of the boundary itself. We denote the lengths of curves L^+ and L^- by $h^+(x)$ and $h^-(x)$. It can be shown that at the interior points of region X the functions $h^{\pm}$ are continuously differentiable and satisfy the conditions /5/

$$\left(\frac{\partial h^{\pm}}{\partial x_1}\right)^2 + \left(\frac{\partial h^{\pm}}{\partial x_2}\right)^2 = \left(\frac{\partial h^{\pm}}{\partial x_3}\right)^2 + \left(\frac{\partial h^{\pm}}{\partial x_4}\right)^2 = 1 \tag{3.1}$$

Equalities (3.1) are the Hamilton–Jacobi equations for certain variational problems on the length of a geodesic. In the problem at hand the length of the geodesic line equals $\min\{h^+, h^-\}$, while the motion time up to capture is

$$S(x) = \min\,[S^+(x),\ S^-(x)], \quad S^{\pm} = h^{\pm} / (1 - \nu) \tag{3.2}$$

Above we assumed that $V(x) = S(x)$, $x \in X_1$, where $V(x)$ is the game value. To make region X_1 more precise we take advantage of the necessary conditions (2.1) and (2.2). On the strength of equalities (3.1) the functions $S^+(x)$ and $S^-(x)$ satisfy the Bellman–Isaacs Eq.(2.2) at the interior points of region X. The function $S(x)$ in (3.2) satisfies this equation outside the points of the hypersurface $\Gamma_0 \subset R^4$:

$$\Gamma_0 = \{x \in X \subset R^4:\ S^+(x) = S^-(x)\} \tag{3.3}$$

When $x \in \Gamma_0$ the function $S(x)$ is differentiable with respect to direction, and $S^{\cdot} = \min[S^{+\cdot}, S^{-\cdot}]$; therefore, conditions (2.1) must be fulfilled at points $x \in \Gamma_0 \cap X_1$. Computations yield

$$\max_{u_j} \min_{u_i} S^{\cdot} = \max_{u_j} \min_{u_i} \min[S^{+\cdot}, S^{-\cdot}] = \qquad (3.4)$$
$$[-1 + \nu R^*(q_3^{\pm}, q_4^{\pm})]/(1-\nu) \leqslant -1$$
$$\min_{u_i} \max_{u_j} S^{\cdot} = \min[-1, R(q^+, q^-)]$$
$$R(q^+, q^-) = [R^*(q_1^{\pm}, q_2^{\pm}) - \nu R^*(q_3^{\pm}, q_4^{\pm})]/(1-\nu)$$
$$R^*(q_1^{\pm}, q_2^{\pm}) = \left\{\frac{1}{2}[1 + (q_1^+ q_1^- + q_2^+ q_2^-)/(1-\nu)^2]\right\}^{1/2}, \quad q^{\pm} = S_x^{\pm}$$

A comparison with (2.1) shows that to region X_1 belongs a part Γ° of surface Γ_0

$$\Gamma^{\circ} = \{x \in X:\ S^+(x) = S^-(x),\ R(q^+, q^-) \geqslant -1\} \qquad (3.5)$$

Surface (3.5) is a dispersal surface /1/; two optimal paths start off from its points, and, as follows from (3.4) and from simple geometric considerations, the direction of obstacle by-pass is chosen by player P. Motion along the geodesic line leads player P at first onto the obstacle's boundary and, after motion along a part of the boundary, emerges into the domain of direct visibility. The points of the border B of surface Γ°

$$B = \{x \in X:\ S^+(x) = S^-(x),\ R(q^+, q^-) = -1\} \qquad (3.6)$$

possess the following geometric property. The vector $q^+ - q^-$ obviously is a normal to surface Γ°. Using relations (3.1), (3.2) and (3.4), we can get that the equalities $(q^+ - q^-, F_q(q^+)) = (q^+ - q^-, F_q(q^-)) = 0$ are fulfilled at the points of set B, signifying, with due regard to (2.3), that the optimal paths "disperse" from the points of border B with tangency to Γ°. We observe that region X_1 abuts that part of the boundary of set X at which constraints (1.2) are vital; thus, these constraints are taken into account by the construction of the primary solution $S(x)$.

Arguing in terms of reverse time /1/, it can be noted that two optimal paths start off from each point of set (3.5). The choice of the path segment entering into the problem's solution is made by the test of the equality of pursuit time. However, for paths arriving onto Γ_0 from the border B of surface Γ° such a selection test is inapplicable by virtue of the necessary conditions (2.1). A similar situation is typical for the existence of an equivocal surface /1/. An attempt to construct a singularity of another type in the given problem is untenable. We give a qualitative description of a discontinuity surface of equivocal type, based on the results in /1,3/. An equivocal surface /1/ is the surface of discontinuity of the partial derivatives of the game's continuous value, and the qualitative behavior of the optimal paths in a neighborhood of the surface is the same as for a switching surface. From the fact that the switching of the optimal control on the equivocal surface obtains for both players simultaneously, it necessarily follows /3/ that singular motions are possible, consisting in the sliding of the phase vector along the discontinuity surface up to going onto its border. Such a sliding is realized when one of the players P or E (defined for the problem being examined) does not switch his control on reaching the singular surface, i.e., uses his "old" control up to the jump. Here the second player is compelled to effect a sliding mode by combining his control up to and after the jump. At any instant of sliding along the singular surface, the first player (who by the terminology in /3/ controls the singular surface) can switch and lead the motion from the singular surface into the primary region. Thus an infinite number of optimal motions issue from the points of the equivocal surface.

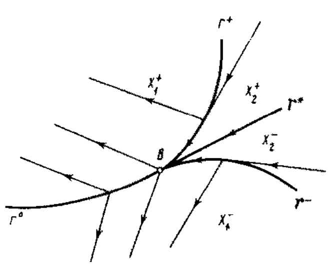

Fig.2

We assume further that the set B of (3.6) is the border (the origin) of two braches Γ^+ and Γ^- of the equivocal surface, in correspondence with the qualitative picture shown in Fig.2. The part of set X, lying between the surfaces Γ^+ and Γ^-, is denoted X_2, $X = X_1 + \Gamma + X_2$, $\Gamma = \Gamma^+ + \Gamma^-$. In region X_1 we have $V(x) = S(x)$; the desired objects now are the surfaces Γ^+, Γ^- and the function $V(x)$, $x \in X_2$. On surfaces Γ^+ and Γ^- are fulfilled, respectively, the equalities $V(x) = S^+(x)$ and $V(x) = S^-(x)$, signify the continuity of the game value $V(x)$. When the discussion touches on both branches Γ^+ and Γ^-, we shall omit the indices in Γ, S and q.

A procedure for constructing the equivocal surface and the game value $V(x)$ in region X_2 was proposed in /3/. Mathematically it is equivalent to solving a certain Cauchy problem with an unknown boundary /6/. Sufficient conditions were obtained in /6/ for the existence and uniqueness in-the-small of the solution of the problem mentioned; in this connection, on the surface there should be prescribed the Cauchy conditions, a constraint in the form of an equality between the vector x and the solution's gradient, and the border of the surface itself. The required surface is constructed from the family of singular paths issuing in reversed time from the border's points /3/.

4. **Equations of the singular characteristics.** The partial derivatives of the game value $V(x)$ on surface Γ have been ambiguously determined. The limit value from region X_1 of the value's gradient on Γ equals $q = S_x$. For the limit value from the region X_2 we retain the original notation $p = V_x$. Having assumed that $p_1^2 + p_2^2 > 0$, $p_3^2 + p_4^2 > 0$ when $x \in \Gamma$, we obtain: the extrema in (2.2) are reached on a unique control vector $u = (u_i, u_j) = F_p$. Then from the results in /3/ it follows that in the problem being analyzed only the equivocal surface is possible, namely, the one covering the optimal paths arriving from region X_2, i.e., these paths are tangent to surface Γ. By virtue of the game value's continuity on Γ the vector $p - q$, $x \in \Gamma$ is a normal to Γ. Then the condition for the tangency, of the optimal path going from X_2, to the surface Γ has the form $(F_p(p), p - q) = 0$ (see (2.3)). This equality is precisely the constraint mentioned at the end of Sect.3. Thus, the equalities

$$V(x) = S(x), \quad (F_p(p), \ p - q) = 0, \quad F(p) + 1 = 0, \quad x \in \Gamma \tag{4.1}$$

are fulfilled on the unknown surface Γ. The Bellman–Isaacs equation $F(p) + 1 = 0$ is fulfilled as well in region X_2. Equalities (4.1) prove sufficient for obtaining the law of variation of vector p along a singular motion, i.e., for deriving the equations of the singular characteristics /3,6/,

$$x^{\cdot} = F_p, \quad p^{\cdot} = -[(S_{xx}F_p, \ F_p)/(F_{pp}q, \ q)]\,(p - q) \tag{4.2}$$

Here S_{xx} and F_{pp} are the matrices of second partial derivatives.

Let us compare system (2.3) and (4.2). In the domain of continuous differentiability of the game value the vectors p and F_p retain, in accord with (2.3), constant values on the optimal paths. Consequently, the motion takes place along a straight line both in the game's phase space as well as in the physical space, viz., in the plane of the players' motion. Equations (4.2) show that the singular paths are, in general, curvilinear. To construct the equivocal surface Γ it is necessary to issue reversed-time solutions of system (4.2) from all points of the border B in (3.6). Here the branch $\Gamma^+ (\Gamma^-)$ corresponds to the quantities S^+, q^+ (S^-, q^-) used in the right-hand sides of system (4.2). To integrate system (4.2) we need to have as well the initial values $p = p(x) = V(x)$ when $x \in B$.

Let us find at first the mentioned initial value of vector p for branch Γ^-. When $x \in B$ there hold the three equalities

$$V(x) - S^+(x) = 0, \quad S^+(x) - S^-(x) = 0, \quad R^c(x) = R(q^+(x), \ q^-(x)) + 1 = 0 \tag{4.3}$$

The dimension of manifold B equals two; therefore, the gradients of the left-hand sides of equalities (4.3), i.e., the vectors $p - q^+$, $q^+ - q^-$ and R_x^c, are linearly dependent in R^4 when $x \in B$. Hence follow two linear equations relative to component of the unknown vector p. Together with the second and third equalities in (4.1) these equations form a fourth-order system relative to vector $p \in R^4$ when $x \in B$. By direct substitution we convince ourselves that the system mentioned has two solutions: $p = q^+$ and

$$p = {}^1\!/_2\,(q^+ + q^-), \quad x \in B \tag{4.4}$$

An analogous consideration for the branch Γ^- also leads to two solutions: $p = q^-$ and (4.4). The fact that a common solution (4.4) exists for both branches Γ^+ and Γ^- signifies that the gradient of the value $V(x)$ can be continuously proceed from X_2 onto the border B. The other solutions $p = q^+$ and $p = q^-$ correspond to the problem's primary solution, which was to be expected. Thus, the quantity (4.4) is the initial value of the adjoint vector in the construction of both branches Γ^+ and Γ^-. When integrating system (4.2) simultaneously with the contruction of surface Γ there is defined on it the field of vector p, i.e., the limit from region X_2 of the value of the game value gradient.

The vector $p - q^+ = (q^- - q^+)/2$, being the normal to Γ^+ when $x \in B$, is directed into the region X_1^+; analogously, the vector $p - q^-$ is directed into X_1^-. Here $X_1^\pm = \{x \in X_1: V(x) = S^\pm(x)\}$. From the results of /3/ it follows that player P controls surface Γ, i.e., when sliding optimally along the surface he has the initiative in switching onto a by-pass of the obstacle (in approaching the primary region). The optimal motions in region X_2 can be constructed by integrating system (2.3) in reverse time with the initial conditions $x = x^0$, $p = p(x^0)$, $x^0 \in \Gamma$. Another method of construction is, when integrating the system (4.2) of the singular characteristics, to set $p^{\cdot} = 0$ by a jump at a certain instant, i.e., to pass to system (2.3). If as such an instant of jump we select the initial instant, we obtain a rectilinear path starting off from a point of border B with the initial value of the adjoint vector (4.4). The collection of such paths, forming a certain surface Γ^*, divides the region X_2 into two subregions $X_2 = X_2^+ + \Gamma^* + X_2^-$. The surface Γ^* touches both the surfaces Γ^+ and Γ^- at the points of border B.

Let us describe the optimal motions starting in X_2. From the points of region X_2^+ (X_2^-) a rectilinear path goes out with tangency onto the surface Γ^+ (Γ^-). At will the player P can at once fall into the primary region X_1^+ (X_1^-) from the point of tangency of the path; to do this the player P must pass (switch) to pursuit along the geodesic by-passing the obstacle (counter-) clockwise. In the opposite case, by using the control optimal for region X_2, player P achieves a curvilinear motion along the surface Γ^*, by which also he can go at any instant into the primary region till he reaches the border B. Having reached set B, player P is obliged to switch to a by-pass of the obstacle along any of two directions. We note that all motions starting in X_2 can lead player P onto manifold B; hitting onto B is inevitable if the motion starts from the points of surface Γ^* or if player P, starting to move from the region $X_2^+(X_2^-)$ wishes, at the final stage, to by-pass the obstacle in the counterclockwise (clockwise) direction.

Let us give a simple geometric interpretation of the tangency condition $(F_p, p-q)=0$ in (4.1). Let φ (respectively, ψ) be the angle between two optimal directions of the velocity u_i of player P (the velocity u_j of player E) on surface Γ. Using relations (3.1) and (3.2), we obtain

$$\cos\varphi = \frac{p_1q_1+p_2q_2}{\sqrt{p_1^2+p_2^2}\sqrt{q_1^2+q_2^2}} = (1-\nu)\frac{p_1q_1+p_2q_2}{\sqrt{p_1^2+p_2^2}}$$

An analogous equality for $\cos\psi$ is obtained by replacing the indices 1,2 by 3,4. Then $(F_p, q) = (-\cos\varphi+\nu\cos\psi)/(1-\nu)$. The function $F(p)$ from (2.2) satisfies the condition $(F_p(p), p) = F(p) = -1$, whence we obtain

$$\cos\varphi - \nu\cos\psi = 1-\nu,\ x\in\Gamma \tag{4.5}$$

Equality (4.5) defines the connection between the angles φ and ψ of the jumps in the directions of the velocities of players P and E.

5. Examples. We consider obstacles of two forms: a circle of unit radius with center at the origin and a segment of the ordinate axis with endpoints $(0,-1)$ and $(0,1)$. For the circle the primary solution of (3.2) is

$$\begin{gathered}S^{\pm}(x) = \frac{1}{1-\nu}\Big(\sqrt{x_1^2+x_2^2-1} - \arccos\frac{1}{\sqrt{x_1^2+x_2^2}} \pm \operatorname{arctg}\frac{x_2}{x_1} + \\ \sqrt{x_3^2+x_4^2-1} - \arccos\frac{1}{\sqrt{x_3^2+x_4^2}} \mp \operatorname{arctg}\frac{x_4}{x_3} + \pi\Big)\end{gathered} \tag{5.1}$$

Formula (5.1) is valid in the region $x_1<0,\ x_3>0$; the passage to the complementary region is effected by a simple transformation in (5.1). The definite symmetry in the problem with a circular obstacle permits us, in general, to restrict ourselves only to the region $x_1<0,\ x_3>0$, and, if desired, to pass to a three-dimensional phase space. The surface of two equal geodesics (3.3) corresponds to the situation when the players are located on an extension of one diameter of the circle, i.e., on a straight line passing through the origin. The equation of this surface is

$$x_2x_3 - x_1x_4 = 0 \tag{5.2}$$

For the case of the segment the formulas analogous to (5.1) and (5.2) are

$$S^{\pm}(x) = \left(\sqrt{x_1^2+(x_2\pm1)^2} + \sqrt{x_3^2+(x_4\pm1)^2}\right)/(1-\nu),\quad S^+(x) = S^-(x) \tag{5.3}$$

The border B of (3.6) is prescribed, for both obstacles, by the equalities

$$S^+(x) = S^-(x),\quad m(x_1,x_2) - \nu m(x_3,x_4) = 1-\nu \tag{5.4}$$

where the function $m(\xi,\eta)$ is

$$\begin{gathered}m(\xi,\eta) = [(\xi^2+\eta^2-1)/(\xi^2+\eta^2)]^{1/2} \\ m(\xi,\eta) = \frac{1}{\sqrt{2}}\left(1+\frac{\xi^2+\eta^2-1}{[\xi^2+(\eta+1)^2]^{1/2}[\xi^2+(\eta-1)^2]^{1/2}}\right)^{1/2}\end{gathered} \tag{5.5}$$

respectively, for the circle and the segment.

Equations (4.2), for $\nu = 1/2$ and for a number of initial points of border B, was integrated numerically, using formulas (5.1)—(5.5). Fig.3 and 4 show typical initials portions of the players' paths in the game's physical space. The players start from the points $P_0^{\pm}$ and

$E_0^{\pm}$ and, moving with maximum velocities, simultaneously reach the points P and E where player P chooses one of two pursuit directions along a geodesic. The superscripts on the initial points delineate those of the surfaces $\Gamma^+, \Gamma^-, \Gamma^*$ on which the corresponding phase path lie in R^4. The positions corresponding to the players' location at points P and E indicated in Figs.3 and 4 lie on the border B of (5.4). The initial portions of the paths leading the position into one and the same point of border B fill the curvilinear triangles $P_0^-PP_0^+$ and $E_0^-EE_0^+$; these portions are two families tangent to the curves PP_0^-, PP_0^+ and EE_0^-, EE_0^+. Between the segments of these two families there is a one-to-one correspondence: to each initial position of player P on the curve $P_0^-P_0^*P_0^+$ there corresponds a position of player E on the segment $E_0^-E_0^*E_0^+$ of the obstacle's boundary, and vice versa. The pair of corresponding points P_0 and E_0 and of the tangent segments starting from them are shown in Fig.3. The players move along the tangents up to the points of tangency, and next, by choice of player P, perform a curvilinear singular motion or pass to a motion along a geodesic.

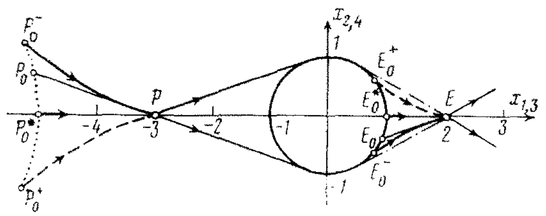

Fig.3

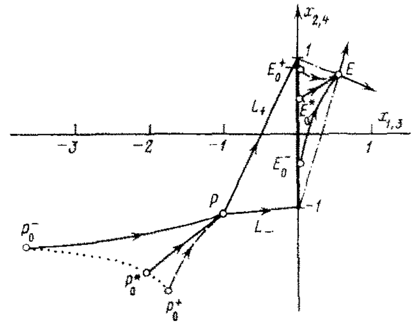

Fig.4

The calculations showed that the curvilinear paths arriving at certain positions of border B differ little from the rectilinear ones, i.e,, for an approximate description of the optimal motions in X we can use only rectilinear paths, not taking into account the motion of player P along the obstacle's boundary. Under such an approximate replacement the above-mentioned family of tangents, say, for the curvilinear triangle $P_0^-PP_0^+$, can be replaced by a bundle of segments joining point P with the points of arc $P_0^-P_0^*P_0^+$. A comparison of the analysis carried out above shows that the picture of the optimal pursuit for the case of an arbitrary obstacle is qualitatively close to the case of the circular obstacle. When passing from a circle to an obstacle of another form there is a loss of definite symmetry of the motions along the surfaces Γ^+ and Γ^-. If to the motion along Γ^* in the case of the circle there corresponds the motion of the players along an extension of the circle's diameter (just as if there was no obstacle), then in the case of an obstacle of another form the rectilinear path segments of players P and E lie, in general, on different straight lines (Fig.4).

We remark that the method proposed in the present paper can be generalized for pursuit problems with an obstacle in space R^n, $n > 2$, if instead of (3.2) we succeed in finding a primary solution of the form

$$S(x) = \min_\alpha S^\circ(x, \alpha), \alpha \in \Omega$$

where Ω is some set of values of parameter α. In the planar case the set Ω consists of two elements.

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